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Special Projective Semi-symmetric Connection on Kenmotsu Manifolds

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Abstract—The aim of the present paper is to study the properties of Kenmotsu manifolds admitting a special projective semi-symmetric connection. We have discovered that if a Kenmotsu manifold of n dimension ($n > 2$), is Ricci flat with respect to special projective semi-symmetric connection whose 1-form η is a recurrent, then the manifold is a certain class of quasi-Einstein manifold. Finally, we show that Weyl projective curvature tensor with respect to special projective semi-symmetric connection is cyclic on the Kenmotsu manifold.

Keywords: Special projective semi-symmetric connection, Kenmotsu manifold, Weyl projective curvature, Tensor.

Mathematics Subject Classification (2020): 53C15; 53C05; 53D15

INTRODUCTION

It is intriguing to examine the properties and invariants of the semi-symmetric metric connection. Friedmann and Schouten (1924) first introduced the idea of semi-symmetric connection. Bartolotti (1930) investigated a geometrical significance for a such connection. Yano (1970) initiated the systematic investigation of a Riemannian manifold with semi-symmetric connection. Further, various geometers had studied semi-symmetric connections in various directions, including Chaubey and Ojha (2012) and Singh *et al*. (2021). Zhao and Song (2001) re-examined and defined a type of semi-symmetric connection on Riemannian manifold which has the same geodesic curves as Riemannian connection D and this new connection is projectively equivalent to D. This is known as projective semi-symmetric connection and it has been further developed by Zhao (2008), Pal *et al*. (2015) and other researchers.

The paper is presented as follows: After introduction, section 2 is devoted to preliminaries. Section 3 provides a brief description of the projectively semi-symmetric connection on the Kenmotsu manifold. Furthermore, in section 4, we have shown that Weyl projective curvature tensor with respect to the special projective semi-symmetric connection is cyclic.

PRELIMINARIES

An n-dimensional Riemannian manifold (M_n, g) of class C^∞ with a 1-form η, the associated vector field ξ and a (1, 1) tensor field ϕ satisfying

$$\phi^2 V_1 + V_1 = \eta(V_1)\xi \tag{2.1}$$

is called an almost contvact manifold and the system (ϕ, ξ, η) is called an almost contact structure to M_n (Mishra, 1984). If the associated Riemannian metric g in M_n satisfy

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2) \tag{2.2}$$

for any arbitrary vector fields V_1, V_2 in M_n, then (M_n, g) is said to be an almost contact metric manifold. Putting ξ for V_1 in (2.2) and using (2.1), we obtain

$$g(\xi, V_2) = \eta(V_2). \tag{2.3}$$

Also $'F(V_1, \xi) = g(\phi V_1, \xi) = \eta(\phi V_1) = 0$ (2.4)

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gives $'F(V_1,V_2)+'F(V_1,V_2)=0$ (2.5)

If moreover,

$$(D_{V_1}\phi)(V_2)=g(\phi V_1,V_2)\xi-\eta(V_2)\phi V_2, \tag{2.6}$$

$$D_{V_1}\xi=V_1-\eta(V_1)\xi \tag{2.7}$$

hold in (M_n, g), where D being the Riemannian connection of g, then (M_n, g) is called a Kenmotsu manifold (Kenmotsu, 1972).

The following relations also hold in a Kenmotsu manifold:

$$K(V_1,V_2)\xi=\eta(V_1)V_2-\eta(V_2)V_1, \tag{2.8}$$

$$K(\xi,V_1)V_2=\eta(V_2)-g(V_1,V_2)\xi, \tag{2.9}$$

$$S(V_1,\xi)=-2n\eta(V_1), \tag{2.10}$$

$$S(V_1,V_2)=g(QV_1,V_2),\ Q\xi=-2n\xi, \tag{2.11}$$

$$(D_{V_1}\eta)(V_2)=g(V_1,V_2)-\eta(V_1)\eta(V_2), \tag{2.12}$$

for arbitrary vector fields $V_1, V_2, V_3 \in T(M_n)$ where S is Ricci tensor and K is the Riemannian curvature tensor with D.

PROJECTIVE SEMI-SYMMETRIC CONNECTION

A linear connection (Yano, 1970) is defined as a semi-symmetric connection $\widetilde{D}$ on an n–dimensional Kenmotsu manifold (M_n, g), if its torsion tensor T with respect to $\widetilde{D}$ is given by

$$T(V_1,V_2)=\widetilde{D}_{V_1}V_2-\widetilde{D}_{V_2}V_1-[V_1,V_2] \tag{3.1}$$

which can also represent as

$$T(V_1,V_2)=A(V_2)V_1-A(V_1)V_2. \tag{3.2}$$

where A is a 1- form associated with a vector field ρ, i.e.,

$$A(V_1)=g(V_1,\rho). \tag{3.3}$$

And this connection $\widetilde{D}$ is a metric connection since it satisfies

$$(\widetilde{D}_{V_1}g)(V_2,V_3)=0. \tag{3.4}$$

$\widetilde{D}$ is said to be a connection projectively equivalent to D if the geodesic with respect to $\widetilde{D}$ are always consistent with those of the connection D on a Riemannian manifold. A semi-symmetric connection which is also projectively equivalent to D, is termed as projective semi-symmetric connection [Zhao and Song, 2001].

Let us consider a projective semi-symmetric connection $\widetilde{D}$ introduced by Zhao and Song (2001) as

$$\widetilde{D}_{V_1}V_2=D_{V_1}V_2+\Psi(V_2)V_1+\Psi(V_1)V_2+\Phi(V_2)V_1-\Phi(V_1)V_2, \tag{3.5}$$

for any vector fields V_1, V_2, where the 1-form ψ and Φ are given below:

$$\Psi(V_1)=\frac{n-1}{2(n+1)}A(V_1)\ \text{and}\ \Phi(V_1)=\frac{1}{2}A(V_1). \tag{3.6}$$

Equations (3.5) and (3.6) give

$$(\widetilde{D}_{V_1}g)(V_2,V_3)=\frac{1}{n+1}\begin{bmatrix}2A(V_1)g(V_2,V_3)-nA(V_2)g(V_1,V_3)\\ -nA(V_3)g(V_1,V_2)\end{bmatrix}. \tag{3.7}$$

It shows that the connection $\widetilde{D}$ given in (3.5) is a metric one. The relations between the curvature tensors $\widetilde{K}$ and K with respect to the projective semi-symmetric connection $\widetilde{D}$ and the Riemannian connection D respectively are given as [Zhao and Song, 2001]

$$\widetilde{K}(V_1,V_2)V_3=K(V_1,V_2)V_3+\alpha(V_1,V_3)V_2-\alpha(V_2,V_3)V_1+\beta(V_1,V_2)V_3, \tag{3.8}$$

where α and β are the tensors of type (0, 2) given by the following relations

$$\alpha(V_1,V_2)=\Psi'(V_1,V_2)+\Phi'(V_2,V_1)-\Psi(V_1)\Phi(V_2)-\Psi(V_2)\Phi(V_1) \tag{3.9}$$

and

$$\beta(V_1,V_2)=\Psi'(V_1,V_2)-\Psi'(V_2,V_1)+\Phi'(V_2,V_1)-\Phi'(V_1,V_2). \tag{3.10}$$

The tensors ψ' and Φ' of type (0, 2) are defined by the following two relations:

$$\Psi'(V_1,V_2)=(D_{V_1}\Psi)(V_2)-\Psi(V_1)\Psi(V_2) \tag{3.11}$$

and

$$\Phi'(V_1,V_2)=(D_{V_1}\Phi)(V_2)-\Phi(V_1)\Phi(V_2). \tag{3.12}$$

A relation between the Ricci tensors of the Kenmotsu manifold with respect to the connections $\widetilde{D}$ and D is obtained by contracting the equation (3.8) over the vector field V_1, is obtained as

$$\tilde{S}(V_2,V_3)=S(V_2,V_3)+\beta(V_2,V_3)-(n-1)\alpha(V_2,V_3). \tag{3.13}$$

Putting $V_2 = V_3 = e_i$ in the above equation, where $\{e_i, 1 \le i \le n\}$, be an orthonormal basis of the tangent space at each point of Kenmotsu manifold and taking summation over i, we get

$$\tilde{r}=r+b-(n-1)a \tag{3.14}$$

where $\tilde{r}=\sum_{i=1}^{n}\tilde{S}(e_ie_i)$, $b=\sum_{i=1}^{n}\beta(e_ie_i)$ and $a=\sum_{i=1}^{n}\alpha(e_ie_i)$.

We identify the 1-form A of the connection $\widetilde{D}$ with the 1-form of the Kenmotsu manifold in order to expand the outview of the projective semi-symmetric connection $\widetilde{D}$ on the manifold. According to the concept of equality between A and η from Equations (3.5) and (3.6), we can write

$$\widetilde{D}_{V_1}V_2 = D_{V_1}V_2 + \frac{1}{2}(c+1)\eta(V_2)V_1 + \frac{1}{2}(c-1)\eta(V_1)V_2, \quad (3.15)$$

where, $c = \frac{n-1}{n+1}$

Now, it can be seen that

$$(\widetilde{D}_{V_1}\eta)(V_2) = (D_{V_1}\eta)(V_2) - c\eta(V_1)\eta(V_2). \quad (3.16)$$

Using the relation $(D_{V_1}\eta)(V_2) = (D_{V_2}\eta)(V_1)$ in the above equation, it gives

$$(\widetilde{D}_{V_1}\eta)(V_2) = (\widetilde{D}_{V_2}\eta)(V_1) \quad (3.17)$$

The connection $\widetilde{D}$ given by the equation (3.5) becomes special projective semi-symmetric connection [Pal *et al.*, 2015]. It can also be verified that for such a projective semi-symmetric connection, the tensor β vanishes and the tensor α is symmetric, i.e.,

$$\beta(V_1, V_2) = 0, \ \alpha(V_1, V_2) = \alpha(V_2, V_1). \quad (3.18)$$

According to these, the curvature tensor, the tensor α, ricci tensor and scalar curvature given by equations (3.8), (3.9), (3.10) and (3.14) become

$$\widetilde{K}(V_1, V_2)V_3 = K(V_1, V_2)V_3 + \alpha(V_1, V_3) - \alpha(V_2, V_3)V_1, \quad (3.19)$$

$$\alpha(V_1, V_2) = \lambda(D_{V_1}\eta)(V_2) - \lambda^2\eta(V_1)\eta(V_2), \quad (3.20)$$

$$\tilde{S}(V_2, V_3) = S(V_2, V_3) - (n-1)\alpha(V_2, V_3), \quad (3.21)$$

and

$$\tilde{r} = r + b - (n-1)a, \quad (3.22)$$

where $\lambda = \frac{1}{2}(c+1)$.

Let us consider a Kenmotsu manifold of dimension-n admitting the special projective semi-symmetric connection $\widetilde{D}$ having 1-form η as a recurrent and satisfies the following relation:

$$(\widetilde{D}_{V_1}\eta)(V_2) = \eta(V_1)\eta(V_2) \quad (3.23)$$

In order of the assumption, the equation (3.20) yields

$$\alpha(V_1, V_2) = (\lambda - \lambda^2)\eta(V_1)\eta(V_2). \quad (3.24)$$

By replacing $V_2 = \xi$ in the above equation, it reduces to the following

$$\alpha(V_1, \xi) = (\lambda - \lambda^2)\eta(V_1) \quad (3.25)$$

Definition 3.1 A non-flat n-dimensional Kenmotsu manifold (M_n, g), $(n > 2)$, is said to be quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies

$$S(V_1, V_2) = \gamma g(V_1, V_2) + \delta\eta(V_1)\eta(V_2) \ \forall \ V_1, V_2 \in T(M_n), \quad (3.26)$$

for smooth functions γ and $\delta(\neq 0)$ where η is a non-zero 1-form associated with Riemannian metric g and associated unit vector field ξ [Chaki and Maithy, 2000].

Theorem 3.1 Let (M_n, g), $(n > 2)$ be an n-dimensional Kenmotsu manifold endowed with a special projective semi-symmetric connection $\widetilde{D}$ whose 1-form η is a recurrent. If the manifold is Ricci flat with respect to $\widetilde{D}$, then it is a certain class of quasi-Einstein manifold.

Proof: Let us suppose that the Kenmotsu manifold (M_n, g), $(n>2)$ is Ricci flat with respect to the special projective semi-symmetric connection $\widetilde{D}$ whose 1-form η is a recurrent, i.e, $\tilde{S} = 0$ and therefore, taking account of equation (3.24), the equation (3.21) gives

$$S(V_2, V_3) = (\lambda - \lambda^2)(n-1)\eta(V_2)\eta(V_3) \quad (3.27)$$

which shows that the manifold is a certain class of quasi-Einstein manifold.

WEYL PROJECTIVE CURVATURE TENSOR OF KENMOTSU MANIFOLD

Theorem 4.1 An n-dimensional Kenmotsu manifold M_n admitting special projective semi-symmetric connection $\widetilde{D}$ satisfy the following relations:

1. $'\widetilde{K}(V_1, V_2, V_3, V_4) = -'\widetilde{K}(V_2, V_1, V_3, V_4)$.
2. $'\widetilde{K}(V_1, V_2, V_3, V_4) \neq {}'\widetilde{K}(V_1, V_2, V_4, V_3)$.
3. $'\widetilde{K}(V_1, V_2, V_3, V_4) \neq {}'\widetilde{K}(V_3, V_4, V_1, V_2)$.
4. $\widetilde{K}(V_1, V_2)V_3 + \widetilde{K}(V_2, V_3)V_1 + \widetilde{K}(V_3, V_1)V_2 = 0$.

Proof: Taking the inner product on both sides of equation (3.19) with respect to V_4, we get

$$'\widetilde{K}(V_1, V_2, V_3, V_4) = K(V_1, V_2, V_3, V_4) + \alpha(V_1, V_3)g(V_2, V_4) - \alpha(V_2, V_3)g(V_1, V_4),$$

for all $V_1, V_2, V_3, V_4, \in T(M_n)$

where $'\widetilde{K}(V_1, V_2, V_3, V_4) = g(\widetilde{K}(V_1, V_2)V_3, V_4)$ and $'K(V_1, V_2, V_3, V_4) = g(K(V_1, V_2)V_3, V_4)$.

In considering equation (3.19) and curvature properties of K, we can easily verify the results (i), (ii) and (iii). With the help of equation (3.19) and Bianchi's first identity, we get

$$\widetilde{K}(V_1, V_2)V_3 + \widetilde{K}(V_2, V_3)V_1 + \widetilde{K}(V_3, V_1)V_2 = 0,$$

which shows that Riemannian curvature tensor with respect to $\widetilde{D}$ satisfies the Bianchi's first identity.

Definition 4.1: The Wely projective tensor W of a Kenmotsu manifold M_n with respect to Riemannian connection D

[Mishra, 1984] is given as

$$W(V_1,V_2)V_3 = K(V_1,V_2)V_3 - \frac{1}{n-1}\{S(V_2,V_3)V_1 - S(V_1,V_3)V_2\}. \quad (4.1)$$

The Weyl projective curvature tensor $\widetilde{W}$ with respect to the projective semi-symmetric connection $\widetilde{D}$ on Kenmotsu manifold is obtained as

$$\widetilde{W}(V_1,V_2)V_3 = \widetilde{K}(V_1,V_2)V_3 - \frac{1}{n-1}\{\tilde{S}(V_2,V_3)V_1 - \tilde{S}(V_1,V_3)V_2\} \quad (4.2)$$

With the help of equation (3.19), the equation (4.2) gives

$$\begin{aligned}\widetilde{W}(V_1,V_2)V_3 &= K(V_1,V_2)V_3 + \alpha(V_1,V_3)V_2 - \alpha(V_2,V_3)V_1 \\ &-\frac{1}{n-1}\{S(V_2,V_3)V_1 - (n-1)\alpha(V_2,V_3)V_1 \\ &-S(V_1,V_3)V_2 + (n-1)\alpha(V_1,V_3)V_2\}. \quad (4.3)\end{aligned}$$

As a consequence of equations (3.18) and (4.1), the above equation yields

$$\widetilde{W}(V_1,V_2)V_3 = W(V_1,V_2)V_3. \quad (4.4)$$

Theorem 4.2 The Ricci tensor of special projective semi-symmetric connection of a Kenmotsu manifold vanishes if and only if $W(V_1,V_2)V_3 = \widetilde{K}(V_1,V_2)V_3$.

Proof: Suppose that

$$W(V_1,V_2)V_3 = \widetilde{K}(V_1,V_2)V_3 \quad (4.5)$$

From (4.1), we have

$$\widetilde{K}(V_1,V_2)V_3 = K(V_1,V_2)V_3 - \frac{1}{n-1}\{S(V_2,V_3)V_1 - S(V_1,V_3)V_2\}.$$

In view of the equation (3.19), the above equation takes the form

$$\{S(V_2,V_3) - (n-1)\alpha(V_2,V_3)\}V_1 = \{S(V_1,V_3) - (n-1)\alpha(V_1,V_3)\}V_2.$$

Using equation (3.21) in the above equation, we get

$$\tilde{S}(V_2,V_3)V_1 = \tilde{S}(V_1,V_3)V_2$$

which is contradiction, gives $\tilde{S}(V_2,V_3) = 0$

Conversely, suppose that $\tilde{S}(V_2,V_3) = 0$, then from the equation (3.12), we have

$$S(V_2,V_3) = (n-1)\alpha(V_2,V_3).$$

Using this in the equation (4.1), we have

$$W(V_1,V_2)V_3 = K(V_1,V_2)V_3 + \alpha(V_1,V_3)V_2 - \alpha(V_2,V_3)V_1,$$

which in view of equation (3.19) gives

$$W(V_1,V_2)V_3 = \widetilde{K}(V_1,V_2)V_3.$$

Theorem 4.3 The Wely-projective curvature tensor $\widetilde{W}$ with respect to special projective semi-symmetric connection $\widetilde{D}$ satisfies

$$\widetilde{W}(V_1,V_2)V_3 + \widetilde{W}(V_2,V_3)V_1 + \widetilde{W}(V_3,V_1)V_2 = 0. \quad (4.6)$$

Proof: Utilizing equations (4.2), (3.19), (3.21) and following Bianchi's first identity of curvature tensor K, we immediately get the result (4.6).

CONCLUSION

The main propose of this work is to study the curvature tensors equipped with special projective semi-symmetric connection on Kenmotsu manifolds. We have found that a Kenmotsu manifold of dimension n, ($n > 2$) admittimg special projective semi-symmetric connection whose 1-form is a recurrent, becomes a certain class of quasi-Einstein manifold if the manifold is Ricci flat with respect to this connection.

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