Generalized 𝔥-almost Ricci Soliton in Almost Kenmotsu Manifold

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Abstract—The primary objective of this paper is to give geometric classifications for 𝔥-almost Ricci soliton associated with almost Kenmotsu manifolds. Initially, we demonstrate that a complete Kenmotsu manifold with a generalized 𝔥-almost Ricci structure is either η-Einstein or its soliton is steady under specific circumstances. Subsequently, it is proved that a Kenmotsu manifold $(k, \varsigma)'$, with $h' \neq 0$, it shows expansion with a generalized 𝔥-almost Ricci metric.

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INTRODUCTION

In recent years, there has been significant attention devoted to the examination of Einstein manifolds and their various extensions within academic circles. Commenced the exploration of Ricci solitons within the field of contact geometry in their research (Sinha and Sharma, 2011). Subsequently, (Tripathi, 2008), along with (Bejan and Crasmareanu, 2011) among others, conducted extensive research on Ricci solitons within contact metric manifolds.

In the later part of the 20th century, Hamilton introduced the concept of Ricci flow. To elaborate, one-parameter family is taken which has a metrics g defined on a Riemannian manifold M^d over an interval $I \subset \mathbb{R}$, where S represents the metric's Ricci tensor g, the equation governing the Ricci flow is as follows:

$$\frac{\partial}{\partial t} g = -2Sg.$$

(Hamilton, 1982) established the existence of a unique solution g to the aforementioned equation for any smooth metric g defined on a compact Riemannian manifold M^d. This solution is defined over some $[0, \in)$, where $\in\ > 0$, which has the condition $g(0) = g_0$.

For the scenario of complete non-compact manifolds, (Shi, 1989) demonstrated the existence of a complete solution under the stipulation that the sectional curvatures of (M^d, g) remain bounded.

As a Riemannian metric, a Ricci soliton satisfies

$$S + \frac{1}{2} L_V g - \lambda g = 0.$$

In the equation provided, The symbol L_V denotes the Lie derivative operator with respect to the vector field V. The tensor S signifies the Ricci tensor corresponding to the metric g, while λ represents a constant. It's evident that a trivial Ricci soliton corresponds to Einstein metric where V can be zero or a vector field that kills. When V is the gradient of a smooth function f, denoted as $V = df$, the soliton is termed a gradient Ricci soliton (Cao, 2010). This characterization enables certain geometers to regard the equation as the defining condition for a Ricci soliton. For further insights into Ricci flow, readers are directed to consult (Chow *et. al.*, 2006).

$$S + \nabla^2 f = \lambda g$$

where S denotes the Ricci tensor of the metric g, and λ is a constant.

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(Gomes *et. al.*, 2017) introduced the concept of 𝔥-almost Ricci soliton for the first time. The authors have conducted a comprehensive study on the properties and characterization of 𝔥-almost Ricci solitons, contributing to the literature on this subject. The study may involve discovering new findings, refining existing theorems, or presenting alternative proofs for known results. The paper also discusses various aspects of 𝔥-almost Ricci solitons, including their geometric interpretations, stability characteristics, classification outcomes, and connections to other geometric constructs. The authors also explore the potential applications of 𝔥-almost Ricci solitons in mathematics and theoretical physics, highlighting their importance in understanding the geometric evolution and curvature properties of Riemannian manifolds. This research offers valuable insights into the theory of 𝔥-almost Ricci solitons and their relevance in contemporary geometers. Also, in a subsequent paper (Ghosh *et. al.*, 2008) studied gradient Ricci soliton of a non-Sasakian (k, ς) - contact manifold. Inspired by these conditions, this paper investigates 𝔥-almost Ricci solitons in the framework of almost Kenmotsu manifolds.

The definition of a 𝔥-almost Ricci soliton, as determined by their findings, is characterized by a complete Riemannian manifold (M^d, g) furnished with a vector field $X_1 \in X_1(M)$, a soliton function $\lambda : M \to \mathbb{R}$, and a function $\mathfrak{h}: M \to \mathbb{R}$. These are all smooth and satisfy the equation as given in (Gomes et. al., 2017).

$$S + \frac{\mathfrak{h}}{2} L_V g = \lambda g$$

For ease of reference, 𝔥-almost Ricci soliton is denoted as $(M^d, g, X_1, \mathfrak{h}, \lambda)$. In the scenario where λ is constant, it is simply termed an 𝔥-Ricci soliton. If $L_V g = L_{\nabla u} g$ holds true for some smooth function $u : M \to \mathbb{R}$, then $(M^d, g, \nabla u, \mathfrak{h}, \lambda)$ is referred to as a gradient 𝔥-almost Ricci soliton having u as the potential function. Under this condition, we can expressed equation (1.2) as:

$$S + \mathfrak{h}\nabla^2 f = \lambda g$$

the notation $\nabla^2 f$ signifies the Hessian of the function f.

The paper is structured as: After preliminaries in section 3, among other results, we establish that if an almost Kenmotsu manifold enable a gradient 𝔥-almost Ricci soliton, then it is a steady Ricci soliton and the manifold admits a Einstein manifold. In this section we also establish that if a Kenmotsu manifold exhibits a non-trivial gradient 𝔥-almost generalized Ricci soliton with a potential vector field is η-Einstein if it is pointwise collinear with the Reeb vector field ξ. As a consequence we obtain that if a Kenmotsu manifold features a non-trivial gradient 𝔥-almost generalized Ricci soliton with vanishing potential vector field and is Einstein, which are denoted by $QX_1 = -2dX_1$ with $\lambda = -m$. In the last section Additionally, we demonstrate that if a $(k, \varsigma)'$ -almost Kenmotsu manifold that $h' \neq 0$ admits a metric of an 𝔥-almost Ricci soliton with $\varphi V=0$, then the soliton is expanding provided $\lambda = \frac{4d^2}{1+d} > 0$.

PRELIMINARIES

A smooth manifold M of dimension $(2d + 1)$ is designated as an almost contact metric manifold provided it accommodates a (1, 1)-tensor field φ, a unit vector field ξ (referred to as the Reeb vector field), and a 1-form, η satisfying the following conditions:

$$\varphi^2 X_1 = -X_1 + \eta(X_1)\xi, \eta(X_1) = g(X_1, \xi) \tag{2.1}$$

for all vector fields X_1 on M. Additionally, a Riemannian metric g is considered an associated (or compatible) metric if it fulfills the condition:

$$g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2) \tag{2.2}$$

for all vector fields X_1, X_2 on M. The combination of an almost contact manifold $M^{2d+1}(\varphi. \xi, \eta)$ with a compatible metric g is referred to as an almost contact metric manifold (Blair, 2010).

An almost Kenmotsu manifold is characterized as an almost contact metric manifold satisfying $d\eta = 0$ and $\Phi = 2\eta \wedge \Phi$, where the fundamental 2-form Φ of the almost contact metric manifold is given by $\Phi(X_1, X_2) = g(X_1, \varphi X_2)$ for any vector fields X_1, X_2 on M (Vanhecke and Janssens, 1981). On the product $M^{2d+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2d+1} and $\mathbb{R}$, an almost complex structure J is defined as:

$$J\left(X_1, f\frac{d}{dt}\right) = \left(\varphi X_1 - f\xi, \eta(X_1)\frac{d}{dt}\right)$$

where t is the coordinate of $\mathbb{R}$, f is a C^∞ function on $M^{2d+1} \times \mathbb{R}$, and X_1 is a vector field tangent to M^{2d+1}. If J is integrable, it is a normal structure in almost contact metric on the manifold M^{2d+1}. A normal almost Kenmotsu manifold is denoted as a Kenmotsu manifold (Kenmotsu, 1972). An almost Kenmotsu manifold transforms into a Kenmotsu manifold if and only if

$$(\nabla_{X_1}\varphi)X_2 = g(\varphi X_1, X_2)\xi - \eta(X_2)\varphi X_1$$

for any vector fields X_1, X_2 on M^{2d+1}. On a Kenmotsu manifold, the following conditions are satisfied (Kenmotsu, 1972):

$$\nabla_{X_1}\xi = X_1 - \eta(X_1)\xi, \tag{2.3}$$

$$R(X_1, X_2)\xi = \eta(X_1)X_2 - \eta(X_2)X_1, \tag{2.4}$$

$$Q\xi = -2d\xi \tag{2.5}$$

for any vector fields X_1, X_2 on M^{2d+1}. In this case, the (1,2) Ricci tensor S has a Ricci operator Q associated with it, where each and every vector field on M^{2d+1}, X_1, X_2 is represented by the notation $S(X_1, X_2) = g(QX_1, X_2)$. R stands for the curvature tensor of g. It has been demonstrated that a Kenmotsu manifold can be locally represented as a warped product $I \times_f N^{2d}$, where I is an open interval with coordinate t, $f = ce^t$ denotes the warping function for some positive constant c, and N^{2d} is a Kählerian manifold (Kenmotsu, 1972).

On an almost Kenmotsu manifold, the following formula holds true: (Dileo and Pastore, 2007), (Dileo and Pastore, 2009):

$$\nabla_{X_1}\xi = \varphi^2 X_1 - \varphi h X_1 \tag{2.6}$$

for any vector field X_1 on M^{2d+1}. We introduce two operators, denoted by h and l, defined as $h = \frac{1}{2}L_\xi\varphi$ and $l = R(., \xi)\xi$, respectively. These operators satisfy the conditions: $h\xi = h'\xi = 0$, $Tr.h = Tr.h' = 0$, $h\varphi = -\varphi h$ where $h' = h.\varphi$, and $Tr.$ denotes the trace operation.

NORMAL ALMOST KENMOTSU MANIFOLD

Within the current section, exploration of Kenmotsu manifolds that exhibit a generalized $\mathfrak{h}$-almost Ricci soliton structure is done. Initially, we give various illustrations of Kenmotsu manifolds accommodating a generalized $\mathfrak{h}$-almost Ricci metric.

Theorem 3.1 *If a gradient $\mathfrak{h}$-almost generalized Ricci soliton is embodied by Kenmotsu metric manifold $M^{2d+1}(\phi,\xi,\eta, g)$, then it either conforms to being η-Einstein or the soliton remains steady.*

Proof. From equation (1.3), we obtain

$$\mathfrak{h}\nabla_{X_1}Df = \lambda X_1 - QX_1 \tag{3.1}$$

Performing the covariant derivative of equation (3.1) with respect to an arbitrary vector field X_2, we derive:

$$\mathfrak{h}\nabla_{X_2}\nabla_{X_1}Df = -\frac{1}{\mathfrak{h}}(\mathfrak{h}X_2)(\lambda X_1 - QX_1) + (\lambda X_2)X_1 + \lambda(\nabla_{X_2}X_1) - (\nabla_{X_2}Q)X_1 - Q(\nabla_{X_2}X_1) \tag{3.2}$$

Employing equations (3.1) and (3.2) in the relation

$$\mathfrak{h}R(X_1,X_2)Df = \mathfrak{h}\nabla_{X_1}\nabla_{X_2}Df - \mathfrak{h}\nabla_{X_2}\nabla_{X_1}Df - \mathfrak{h}\nabla_{[X_1,X_2]}Df$$

we derive

$$\mathfrak{h}R(X_1,X_2)Df = -\frac{1}{\mathfrak{h}}(X_1\mathfrak{h})(\lambda X_2 - QX_1) + (\lambda X_1)X_2 - (\nabla_{X_1}Q)X_2 + \frac{1}{\mathfrak{h}}(X_2\mathfrak{h})(\lambda X_1 - QX_1) - (\lambda X_2)X_1 + (\nabla_{X_2}Q)X_1 \tag{3.3}$$

By taking the inner product of equation (3.3) with ξ and utilizing equation (2.5), we obtain:

$$g(\mathfrak{h}R(X_1,X_2)Df,\xi) = (X_1\lambda)\eta(X_2) - (X_2\lambda)\eta(X_1) - g\left((\nabla_{X_2}Q)X_1,\xi\right) - g((\nabla_{X_1}Q)X_2,\xi) + \frac{1}{\mathfrak{h}}(\lambda+2d)[(X_2\mathfrak{h})\eta(X_1) - (X_1\mathfrak{h})\eta(X_2)]. \tag{3.4}$$

Again, taking the inner product of equation (2.4) with Df and substituting it into the equation (3.4), we get:

$$\frac{1}{\mathfrak{h}}(\lambda+2d)[(X_2\mathfrak{h})(\eta(X_1) - (X_1\mathfrak{h})\eta(X_2)] - (X_2\lambda)\eta(X_1) + g\left((\nabla_{X_1}Q)X_2,\xi\right) + (X_1\lambda)\eta(X_2) + g\left((\nabla_{X_2}Q)\xi,X_1\right) - \eta(X_2)g(X_1,Df) + \mathfrak{h}\eta(X_1)g(X_2,Df)] = 0. \tag{3.5}$$

Substituting X_2 with ξ in equation (3.5) and utilizing the relation $(\nabla_\xi)X_2 = -2QX_2 - 4dX_2$ (see Lemma 2 of (Ghosh, 2019)) we have

$$[\sigma(\xi h) - h(\xi\lambda) + h^2(\xi f)]\eta(X_1) = \sigma(X_1 h) - h(X_1\lambda) + h^2(X_1 f) \tag{3.6}$$

where $\sigma = \lambda + 2d$. Contracting equation (3.3) along an arbitrary vector field X_1 results in

$$\mathfrak{h}S(Df,X_2) = \frac{1}{\mathfrak{h}}S(X_2,D\mathfrak{h}) - 2d(X_2\lambda) + \frac{1}{2}(X_2 r) + \frac{1}{\mathfrak{h}}(2d\lambda - r) \tag{3.7}$$

Substituting X_2 with ξ and employing equation (2.5) in equation (3.7), we obtain:

$$-2d\mathfrak{h}^2(\xi f) - (2d\sigma - (r + 2d(2d+1))(\xi\mathfrak{h}) + 2d\mathfrak{h}(\xi\lambda) - \frac{1}{2}\mathfrak{h}(\xi r) = 0. \tag{3.8}$$

Moreover, on the Kenmotsu manifold, we have $\xi r = -2(r+2d)(2d+1))$ (Lemma 2 of (Ghosh, 2019)). Using into the above equation yields:

$$2d[\sigma(\xi\mathfrak{h}) - \mathfrak{h}(\xi\lambda) + \mathfrak{h}^2(\xi f)] = r + 2d(2d+1)(\mathfrak{h} + \xi\mathfrak{h}). \tag{3.9}$$

Substituting X_2 with ξ in equation (3.3) and employing $\mathfrak{h}R(X_1,\xi)X_2 = \mathfrak{h}g\xi - \mathfrak{h}\eta(X_2)X_1$, we get

$$\frac{1}{\mathfrak{h}}[\sigma(X_1\mathfrak{h}) - \mathfrak{h}(X_1\lambda) + \mathfrak{h}^2(X_1 f)]\xi = \mathfrak{h}\eta(\xi f)X_1 - (\xi\lambda)X_1 + \frac{1}{\mathfrak{h}}(\lambda X_1)(\xi\mathfrak{h}) - \left(1 + \frac{\xi\mathfrak{h}}{\mathfrak{h}}\right)QX_1 - 2dX_1. \tag{3.10}$$

Combining equations (3.6), (3.9), and (3.10), we derive the following relation:

$$2d[\mathfrak{h}(\xi f)X_1 + \lambda X_1/\mathfrak{h}(\xi\lambda) - (\xi\lambda)X_1 - \left(1 + \frac{\xi\mathfrak{h}}{\mathfrak{h}}\right)QX_1 - 2dX_1] = \frac{1}{\mathfrak{h}}(r + 2d(2d+1))(\mathfrak{h} + \xi\mathfrak{h})\eta(X_1)\xi. \tag{3.11}$$

Suppose $\mathfrak{h}$ is constant and $\xi\mathfrak{h} = 0$. Utilizing equation (3.9), we have:

$$\big(r + 2d(2d+1)\big)\eta(X_1)\xi = -\frac{2d\lambda}{\mathfrak{h}}(\xi\lambda)X_1 + \big(r + 2d(2d+1)\big)X_1$$
$$-2dQX_1 - 4d^2X_1.$$

Let $X_1 = \xi$ and $\lambda(\xi\lambda) = 0$. If $\lambda = 0$, the soliton remains steady. If $\xi\lambda = 0$, then the manifold is η-Einstein. This brings the proof to a close.

Theorem 3.2 *A Kenmotsu manifold is η-Einstein if it has a non-trivial gradient $\mathfrak{h}$-almost generalized Ricci soliton for which the potential vector field is pointwise collinear with the Reeb vector field ξ.*

Proof. Assume V which represents a potential vector field and the Reeb vector field ξ are pointwise collinear, such that $V = F\xi$, in which the smooth function F. By performing a covariant differentiation along an arbitrary vector field X_1 of $V = F\xi$ and employing equation (2.3), we obtain:

$$\nabla_{X_1}V = (X_1F)\xi + F(-\varphi^2X_1 - \varphi\mathfrak{h}X_1). \tag{3.12}$$

Substituting equation (3.12) into equation (1.2) yields:

$$\frac{\mathfrak{h}}{2}[(X_1F)\eta(X_2) + (X_2F)\eta(X_1)] + \mathfrak{h}Fg(\mathfrak{h}'X_1, X_2)$$
$$-\mathfrak{h}F\eta(X_1)\eta(X_2) + S(X_1,X_2) = (\lambda - \mathfrak{h}F)g(X_1,X_2) \tag{3.13}$$

for any vector fields X_1 and X_2.

Substituting X_1, X_2 with ξ in equation (3.13) and employing equation (2.5), we get

$$\xi F = \frac{1}{\mathfrak{h}}(\lambda + 2d).$$

Furthermore, substituting X_2 with ξ and employing the final expression in equation (3.13), we have

$$X_1F = \frac{1}{\mathfrak{h}}(\lambda + 2n)\eta(X_1). \tag{3.14}$$

Contracting equation (3.13) and inserting it into the preceding equation (3.14), we have

$$r = 2(\lambda - \mathfrak{h}F - 1). \tag{3.15}$$

As a consequence of equations (3.14) and (3.15), Equation (3.13) simplifies to the following form:

$$QX_1 = \left(\frac{r}{2d} + 1\right)X_1 - \left(\frac{r}{2d} + 2d + 1\right)\eta(X_1)\xi \tag{3.16}$$

For any vector field X_1, thereby establishing the manifold as η-Einstein.

Assume that F is constant. From equation (3.14), it follows that $\lambda = -2d$. Substituting this value into equation (3.15), we deduce a constant r. Consequently, ξr vanishes, implying $r = -2d(2d + 1)$. Replacing r and λ with their respective values into equation (3.15), we find $F = 0$, which further implies $\lambda = -2d$.

Theorem 3.3 *If a Kenmotsu manifold features a non-trivial gradient $\mathfrak{h}$-almost generalized Ricci soliton with a vanishing potential vector field, then it is Einstein, denoted by $QX_1 - 2dX_1$ with $\lambda = -m$.*

NON-NORMAL ALMOST KENMOTSU MANIFOLD

If ξ, that Reeb vector field, is contained within the generalized $(k. \varsigma)$-nullity distribution, then the almost Kenmotsu manifold $M^{(2d+1)}$ is refered to as a generalized almost Kenmotsu manifold (k, ς). i.e.,

$$k[\eta(X_2)X_1 - \eta(X_1)X_2] + \varsigma[\eta(X_2)hX_1 - \eta(X_1)hX_2] = R(X_1,X_2)\xi. \tag{4.1}$$

The condition holds true for all vector fields X_1, X_2 on M, with k and ς representing smooth functions defined on M. An almost Kenmotsu manifold $M^{(2d+1)}$ is termed a generalized $(k, \varsigma)'$-almost Kenmotsu manifold if the Reeb vector field ξ is part of the generalized $(k. \varsigma)'$-nullity distribution, which is defined as:

$$k[\eta(X_2)X_1 - \eta(X_1)X_2] + \varsigma[\eta(X_2)h'X_1 - \eta(X_1)h'X_2] = R(X_1,X_2)\xi \tag{4.2}$$

this condition applies to all vector fields X_1, X_2 on M, where k, ς denote smooth functions defined on M, and $h'=h\circ\varphi$ (Dileo and Pastore, 2009). Furthermore, if both k and ς are constants in equation (4.2), then M is referred to as a $(k, \varsigma)'$-almost Kenmotsu manifold (Dileo and Pastore,2009), (Wang and Liu, 2016), (Khatri and Singh, 2023). The following relations hold on a generalized (k, ς) or $(k, \varsigma)'$-almost Kenmotsu manifold with $h \neq 0$ (equivalently, $h' \neq 0$):

$$h'^2 = (k+1)\varphi^2, \quad h^2 = (k+1)\varphi^2, \tag{4.3}$$

$$Q\xi = 2dk\xi. \tag{4.4}$$

From equation (4.3), it follows that $k \leq -1$ and $v = \pm\sqrt{(-k-1)}$, where v represents an eigenvalue corresponding to the eigenvector $X_1 \in D\big(D = Ker(\eta)\big)$ of h'. The equality holds if and only if $h = 0$ (equivalently, $h' = 0$). Therefore, $h' \neq 0$ if and only if $k < -1$.

Lemma 4.1 *(Wang and Liu, 2016) Consider $M^{2d+1}(\varphi, \xi, \eta, g)$ to be a generalized $(k, \varsigma)^1$-almost Kenmotsu manifold with $h' \neq 0$. For $d > 1$, the Ricci operator Q of M can be expressed as:*

$$-2dX_1 + 2d(k+1)\eta(X_1)\xi - [\varsigma - 2(d-1)h']X_1 = QX_1$$

for any vector field X_1 on M. Additionally, if k and ς are constants and $d \geq 1$, then $\varsigma = -2$, and thus

$$QX_1 = -2dX_1 + 2d(k+1)\eta(X_1)\xi - 2dh'X_1 \tag{4.5}$$

The scalar curvature of M for any vector field X_1 on M is $2d(k - 2d)$ in both cases.

Theorem 4.2 *A soliton is said to be expanding if the metric of a (k, ς)′-almost Kenmotsu manifold with h′ ≠ 0 admits a* $\mathfrak{h}$*-almost Ricci soliton with* $\varphi V = 0$.

Proof. Given the hypothesis $\varphi V = 0$, applying the operator φ yields $V = \eta(V)\xi$, that is, $V = F\xi$, where the function F is smooth. Equation (1.2) is obtained by calculating the final equation's covariant derivative along any vector field X_1 and adding it:

$$S(X_1, X_2) + \frac{\mathfrak{h}}{2}[(X_1F)\eta(X_2) + (X_2F)\eta(X_1)] - Fg(h'^{X_1}, X_2)$$
$$-\mathfrak{h}F\eta(X_1)\eta(X_2) = (\lambda - \mathfrak{h}F)g(X_1, X_2). \tag{4.6}$$

Substituting X_1 with ξ in equation (4.6) yields

$$\frac{\mathfrak{h}}{2}(X_2F) = \left(\lambda + 2dk - \frac{\mathfrak{h}}{2}(\xi F)\right)\eta(X_2) \tag{4.7}$$

for any vector field X_1 on M. Contracting equation (4.4) and utilizing a lemma, we obtain:

$$\mathfrak{h}(\xi F) = (2d+1)(\lambda - 2\mathfrak{h}F) - \mathfrak{h}F(k+2) - \left(1 - \frac{\mathfrak{h}F}{2d}\right)(2d(k-2d)). \tag{4.8}$$

Substituting $X_2$with ξ in equation (4.7) and combining it with equation (4.8) gives $F = \frac{d(\lambda+2d)}{\mathfrak{h}(3d+1)}$. By inserting equation (4.7) into equation (4.6) and utilizing it in the lemma, we derive

$$\left[\frac{d(\lambda+2+8d)}{(3d+1)}\mathfrak{h}X_1 + (\lambda - 2\mathfrak{h}F)\varphi X_1\right](k+1) = 0. \tag{4.9}$$

Substituting X_1 with $h'X_1$ in equation (4.9) and considering $h' \neq 0$, we obtain $\lambda = \frac{4d^2}{1+d} > 0$. This concludes the proof, establishing that the soliton is expanding.

CONCLUSION

The study investigates almost Kenmotsu manifolds with gradient $\mathfrak{h}$-almost Ricci solitons. It concluded that if a gradient is permitted by a manifold, $\mathfrak{h}$-almost Ricci soliton, it must be an Einstein manifold, and the Ricci soliton must be steady. The analysis also showed that a Kenmotsu manifold with a non-trivial gradient $\mathfrak{h}$-almost generalized η-Einstein is required for a Ricci soliton with a potential vector field collinear with the Reeb vector field. Additionally, if a manifold has a non-trivial gradient $\mathfrak{h}$-almost generalized Ricci soliton with a vanishing potential vector field, Einstein manifold, that is. The study also found that the presence of an $\mathfrak{h}$-almost Ricci soliton with certain conditions leads to an expanding soliton. These findings contribute to our understanding of almost Kenmotsu manifolds and their associated solitons.

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