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Refinements of Generalized Jacobi Method: A Higher-Order Approach

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Abstract—This study presents a comparative analysis of the p^{th} Refinement of Generalized Jacobi method, focusing on its derivation, convergence properties, and numerical performance. We first derive the Third Refinement of Generalized Jacobi method and establish a general formula applicable to any p^{th} Refinement. The analysis rigorously proves convergence for various types of matrices, including Strictly Diagonally Dominant (SDD) and M-matrices. Additionally, we demonstrate that the convergence rate of the $(p + 1)^{th}$ Refinement surpasses that of the p^{th} Refinement, assuming the Generalized Jacobi method converges. Numerical examples are provided to support our theoretical findings, showcasing the improved convergence rates of the $(p + 1)^{th}$ Refinement compared to the p^{th} Refinement. These results underline the advantages of higher-order refinements in the Generalized Jacobi method.

Keywords: System of Linear Equations; Iterative methods; Jacobi; Generalized Jacobi; SDD matrix; M-matrix, Banded matrix.

MR(2020) Subject Classification: 65F10, 65F15, 65F50.

INTRODUCTION

The Jacobi method, introduced by Carl Gustav Jacob Jacobi in the 19th century, is a fundamental iterative approach for solving systems of linear equations. Despite its simplicity and effectiveness in solving such systems, it often suffers from slow rate of convergence, particularly for matrices that are diagonally dominant or possess closely clustered eigenvalues. To address these limitations, various refinement techniques have been developed over time.

Salkuyeh (2007) proposed an extension of the Jacobi method that incorporates the concept of a banded matrix, which significantly improves convergence for SDD and M-matrices (Salkuyeh 2007). Based on this work, Dafchahi (2008) introduced the Refinement of Jacobi (RJ) method, which accelerates convergence for systems with SDD matrices by employing a new iterative update scheme. This method has demonstrated convergence for Symmetric Positive Definite (SPD) and M-matrices (Dafchahi 2008).

Vatti and Gonfa (2011) further extended the method by combining generalization and refinement techniques to create the Refinement of Generalized Jacobi method. Their study demonstrated that this method is more efficient than the Refinement of Jacobi method, particularly in terms of performance, convergence speed, storage requirements, and accuracy (Vatti and Gonfa 2011).

Further advancements were made by Enyew *et al*. (2019), who introduced the Second Refinement of Jacobi (SRJ) method, enhancing convergence for SDD, SPD, and M-matrices. This method reduced the number of iterations required and lowered the spectral radius, thus improving the rate of convergence (Enyew *et al*. 2019). In 2020, Enyew *et al*. introduced the Second Refinement of Generalized Jacobi (SRGJ) method, which showed superior performance in terms of iteration count and spectral radius compared to traditional Jacobi and related methods, with guaranteed convergence for SDD, SPD, and M-matrices (Enyew *et al*. 2020).

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This paper presents the development of higher-order refinements of the Generalized Jacobi method and investigates their convergence properties across various types of matrices. Section 2, provides the necessary preliminary definitions, theorems, and a review of existing methods. In Section 3, we derive Third Refinement of Generalized Jacobi method and extend the approach to higher-order refinements. Additionally, we explore the convergence properties of these higher-order methods and provide numerical examples that illustrate their efficiency in comparison to other refinement techniques. Finally, Section 4 summatrizes the key findings and offers concluding remarks on the overall performance of the proposed methods.

MATERIALS AND METHODS

Let $A = (a_{ij})$ be an $n \times n$ matrix. Then $D_m = (d_{ij})$ is the banded matrix of bandwidth $2m + 1$, defined as follows:

$$d_{ij} = \begin{cases} a_{ij}, & |i-j| \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

Let us consider the splitting of matrix A as

$$A = D_m + L_m + U_m$$

where L_m and U_m are the strictly lower and strictly upper parts of the matrix $A - D_m$, respectively.

Thus, we have,

$$D_m = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m+1} & & & \\ \vdots & \ddots & & \ddots & & \\ a_{m+1,1} & & \vdots & & & \\ & & \vdots & & & a_{n-m,n} \\ & \ddots & & & \ddots & \vdots \\ & & & a_{n,n-m} & \cdots & a_{n,n} \end{pmatrix}$$

$$L_m = \begin{pmatrix} & & & \\ & & & \\ a_{m+2,1} & & & \\ \vdots & \ddots & & \\ a_{n,1} & \cdots & a_{n,n-m-1} & \end{pmatrix},$$

$$U_m = \begin{pmatrix} & & a_{1,m+2} & \cdots & a_{1,n} \\ & & & \ddots & \vdots \\ & & & & a_{n-m-1,n} \\ & & & & \\ & & & & \end{pmatrix}$$

Definition 2.1. Spectral Radius of a mattrix:(Butt, 2015) *The spectral radius of a matrix A is defined as the largest absolute value of its eigenvalues. That is*

$$\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|,$$

where λ_i are the eigenvalues of A.

Definition 2.2. Strictly Diagonally Dominant (SDD) matrix: (Salkuyeh, 2007) *A square matrix $A = (a_{ij})$ is said to be Strictly Diagonally Dominant (SDD) if*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad \text{for all } i = 1, 2, \ldots, n,$$

where a_{ij} represents the elements of the matrix.

Definition 2.3. Symmetric Positive Definite (SPD) matrix: (Salkuyeh, 2007) *An $n \times n$ matrix $A = (a_{ij})$ is said to be Symmetric Positive Definite if A is symmetric ($A = A^T$) and positive definite, that is, $x^T Ax > 0$ for all $x \neq 0$.*

Definition 2.4. M-matrix: (Enyew *et al.*, 2020) *A matrix is classified as an M-matrix if it adheres to the following criteria:*

- $a_{ii} > 0$ for $i = 1, 2, \ldots, n$.
- $a_{ij} \leq 0$ for $i \neq j$ and $i, j = 1, 2, \ldots, n$.
- A is non-singular.
- $A^{-1} \geq 0$.

Lemma 2.1. (Enyew *et al.*, 2020) *The Spectral radius satisfies the following rule:*

$$\rho(A^k) = (\rho(A))^k \quad \text{for all} \quad k \in N \quad \text{and} \quad A \in C^{n \times n}.$$

Theorem 2.1. (Salkuyeh, 2007) *For any initial approximation $x^{(0)} \in \mathbb{R}$, the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ of approximations defined by*

$$x^{(k+1)} = Tx^{(k)} + c,$$

for each $k \geq 0$, and $c \neq 0$, converges to the unique solution of $x = Tx + c$, if and only if $\rho(T) < 1$. Also, since $\rho(T) \leq ||T||$ for any natural norm, $\rho(T) < 1$ is equivalent to $||T|| < 1$.

Now we will discuss the established Jacobi method and its refinement forms.

For the system of linear equations,

$$Ax = b \tag{1}$$

where $A = (a_{ij})$ is a non-singular real matrix of order n, b is a real n-dimensional vector, and x is the vector to be determined.

Let us consider the splitting of matrix A as,

$$A = D_m + L_m + U_m$$

where D_m is the banded matrix of bandwidth $2m + 1$, and L_m and U_m are the strictly lower and strictly upper parts of the matrix $A - D_m$, respectively. Then,

- The Generalized Jacobi method is given by (Salkuyeh, 2007):

$$x^{(k+1)} = -D_m^{-1}(L_m + U_m)x^{(k)} + D_m^{-1}b, \tag{2}$$

  where $k = 0, 1, 2, \ldots$, and $m = 0, 1, 2, \ldots$

- The Refinement (or First Refinement) of Generalized Jacobi method is given by (Dafchahi, 2008):

$$x^{(k+1)} = \left(-D_m^{-1}(L_m + U_m)\right)^2 x^{(k)} + \left(I - D_m^{-1}(L_m + U_m)\right) D_m^{-1}b, \tag{3}$$

  where $k = 0, 1, 2, \ldots$, and $m = 0, 1, 2, \ldots$

- The Second Refinement of Generalized Jacobi method is given by (Enyew *et al*., 2019):

$$x^{(k+1)} = \left(-D_m^{-1}(L_m + U_m)\right)^3 x^{(k)} + \left(I - D_m^{-1}(L_m + U_m) + \left(-D_m^{-1}(L_m + U_m)\right)^2\right) D_m^{-1}b, \tag{4}$$

  where $k = 0, 1, 2, \ldots$, and $m = 0, 1, 2, \ldots$

Theorem 2.2. (Salkuyeh, 2007) *Let A represent an SDD and M-matrix. Then, for any natural number m ≤ n, the Generalized Jacobi (GJ) method exhibits convergence with an initial guess* $x^{(0)}$.

Theorem 2.3. (Vatti and Gonfa, 2011; Enyew *et al*., 2019) *Let A represent an SDD and M-matrix. Then, for any natural number m ≤ n, the Refinement Generalized Jacobi (RGJ) method exhibits convergence with an initial guess* $x^{(0)}$.

Theorem 2.4. (Enyew *et al*., 2019) *Let A represent an SDD and M-matrix. Then, for any natural number m ≤ n, the Second Refinement Generalized Jacobi (SRGJ) method exhibits convergence with an initial guess* $x^{(0)}$.

RESULTS AND DISCUSSION

Consider the system (1)

$$Ax = b$$
$$\Rightarrow (D_m + L_m + U_m)x = b$$
$$\Rightarrow D_m x = b - (L_m + U_m)x$$
$$\Rightarrow D_m x = b + (D_m - A)x$$
$$\Rightarrow x = x + D_m{}^{-1}(b - Ax)$$

So, the Refinement iterative form will be,

$$x^{(k+1)} = \tilde{x}^{(k+1)} + D_m{}^{-1}(b - A\tilde{x}^{(k+1)}) \tag{5}$$

To find the desired Third Refinement of Generalized Jacobi, we put the value of $\tilde{x}^{(k+1)} = (-D_m{}^{-1}(L_m + U_m))^3 x^{(k)} + (I - D_m{}^{-1}(L_m + U_m) + (-D_m{}^{-1}(L_m + U_m))^2)D_m{}^{-1}b$ taking from (4) in the equation (5). So, we get,

$$\begin{aligned} x^{(k+1)} =& (-D_m{}^{-1}(L_m + U_m))^4 x^{(k)} + (I + (-D_m{}^{-1}(L_m + U_m)) \\ &+ (-D_m{}^{-1}(L_m + U_m))^2)D_m{}^{-1}b + \\ &D_m{}^{-1}(b - A((-D_m{}^{-1}(L_m + U_m))^3 x^{(k)} + (I - D_m{}^{-1}(L_m + U_m) \\ &+ (-D_m{}^{-1}(L_m + U_m))^2)D_m{}^{-1}b)) \end{aligned}$$

After some simplification and arrangement, finally we get,

$$\begin{aligned} x^{(k+1)} =& (-D_m{}^{-1}(L_m + U_m))^4 x^{(k)} + (I + (-D_m{}^{-1}(L_m + U_m)) \\ &+ (-D_m{}^{-1}(L_m + U_m))^2 + (-D_m{}^{-1}(L_m + U_m))^3)D_m{}^{-1}b \end{aligned}$$

where, $k = 0, 1, 2, \ldots$, and $m = 0, 1, 2, \ldots$

Which is the required Third Refinement of Generalized Jacobi method.

In the same way, we can also find the Fourth Refinement of Generalized Jacobi method and it is,

$$\begin{aligned} x^{(k+1)} =& (-D_m{}^{-1}(L_m + U_m))^5 x^{(k)} + (I + (-D_m{}^{-1}(L_m + U_m)) \\ &+ (-D_m{}^{-1}(L_m + U_m))^2 + (-D_m{}^{-1}(L_m + U_m))^3 \\ &+ (-D_m{}^{-1}(L_m + U_m))^4)D_m{}^{-1}b \end{aligned}$$

where, $k = 0, 1, 2, \ldots$, and $m = 0, 1, 2, \ldots$

By generalizing the above sequence we can talk about any p^{th} Refinement of Generalized Jacobi method, for $p = 0, 1, 2, \ldots$ and it will be of the form,

$$\begin{aligned} x^{(k+1)} =& (-D_m{}^{-1}(L_m + U_m))^{p+1} x^{(k)} + (I + (-D_m{}^{-1}(L_m + U_m)) \\ &+ (-D_m{}^{-1}(L_m + U_m))^2 + (-D_m{}^{-1}(L_m + U_m))^3 + \ldots \\ &+ (-D_m{}^{-1}(L_m + U_m))^p)D_m{}^{-1}b \end{aligned}$$

where, $k = 0, 1, 2, \ldots$, and $m = 0, 1, 2, \ldots$

If $p = 0$, then the scheme will be the Generalized Jacobi method.

Also, if $p = 0$ and $m = 0$, it is nothing but the original Jacobi method.

Now, we will discuss the convergence of the p^{th} Refinement of Generalized Jacobi method.

Theorem 4.1. *For an SDD matrix, any* p^{th} *Refinement of Generalized Jacobi method is convergent for any initial approximation* $x^{(0)}$.

Proof. From Theorem 2.2 Generalized Jacobi method is convergent.

So the spectral radius of the iteration matrix is less than 1. That is,

$$\rho(-D_m{}^{-1}(L_m + U_m)) < 1. \quad (6)$$

Since the Spectral radius of p^{th} Refinement of Generalized Jacobi method is,

$\rho((-D_m{}^{-1}(L_m + U_m))^{p+1})$, where $p = 1, 2, ...$

Now by using Lemma 2.1 and equation 6,

$\rho((-D_m{}^{-1}(L_m + U_m))^{p+1}) = (\rho(-D_m{}^{-1}(L_m + U_m)))^{p+1} < 1.$

So for the SDD matrix p^{th} Refinement of Generalized Jacobi Method is convergent for any initial guess.

Theorem 4.2. *If the coefficient matrix is an M-matrix, then any p^{th} Refinement of Generalized Jacobi method is convergent for any initial guess $x^{(0)}$.*

Proof. Similar to the above theorem, by using Theorem 2.2 and Lemma 2.1 we can prove it.

Theorem 4.3 *If the Generalized Jacobi method is convergent, then any p^{th} Refinement of the Generalized Jacobi method converges faster than the Generalized Jacobi method. Additionally, any $(p + 1)^{th}$ Refinement of the Generalized Jacobi method converges faster than any p^{th} Refinement of the Generalized Jacobi method for any initial guess $x^{(0)}$.*

Proof. We can write, GJ method as,

$$x^{(k+1)} = Gx^{(k)} + K_1.$$

RGJ (FRGJ) method as,

$$x^{(k+1)} = G^2x^{(k)} + K_2.$$

SRGJ method as,

$$x^{(k+1)} = G^3x^{(k)} + K_3.$$

TRGJ method as,

$$x^{(k+1)} = G^4x^{(k)} + K_4.$$

$\vdots$

Any p^{th} RGJ method as,

$$x^{(k+1)} = G^{(p+1)}x^{(k)} + K_{p+1}.$$

Where,

$G = -D_m{}^{-1}(L_m + U_m).$

$K_1 = D_m{}^{-1}b.$

$K_2 = D_m{}^{-1}b(I + (-D_m{}^{-1}(L_m + U_m))).$

$K_3 = D_m{}^{-1}b(I + (-D_m{}^{-1}(L_m + U_m)) + (-D_m{}^{-1}(L_m + U_m))^2).$

$$K_4 = D_m{}^{-1}b(I + (-D_m{}^{-1}(L_m + U_m)) + (-D_m{}^{-1}(L_m + U_m))^2 + (-D_m{}^{-1}(L_m + U_m))^3).$$

$$K_{p+1} = D_m{}^{-1}b(I + (-D_m{}^{-1}(L_m + U_m)) + (-D_m{}^{-1}(L_m + U_m))^2 + ... + (-D_m{}^{-1}(L_m + U_m))^p).$$

Given that, the GJ method is convergent.

So,

$$||G|| < 1 \quad (7)$$

Let x be the exact solution of $Ax = b$, then

$x = Gx + K_1, \quad x = G^2x + K_2, \quad x = G^3x + K_3, \quad x = G^4x + K_4,$

$x = G^{p+1}x + K_{p+1}.$

Now, consider the GJ method,

$$\begin{aligned}
&x^{(k+1)} = Gx^{(k)} + K_1\\
\Rightarrow &x^{(k+1)} - x = Gx^{(k)} - x + K_1\\
\Rightarrow &x^{(k+1)} - x = G(x^{(k)} - x) + Gx + K_1 - x\\
\Rightarrow &x^{(k+1)} - x = G(x^{(k)} - x)\\
\Rightarrow &||x^{(k+1)} - x|| = ||G(x^{(k)} - x)||\\
\Rightarrow &||x^{(k+1)} - x|| \leq ||G||\ ||(x^{(k)} - x)||\\
&\leq ||G^2||\ ||(x^{(k-1)} - x)||\\
&\leq ... \leq ||G^{k+1}||\ ||(x^{(0)} - x)||\\
&= ||G||^{k+1}\ ||(x^{(0)} - x)||
\end{aligned}$$

Now, considering RGJ,

$$\begin{aligned}
&x^{(k+1)} = G^2x^{(k)} + K_2\\
\Rightarrow &x^{(k+1)} - x = G^2x^{(k)} - x + K_2\\
\Rightarrow &x^{(k+1)} - x = G^2(x^{(k)} - x) + G^2x + K_2 - x\\
\Rightarrow &x^{(k+1)} - x = G^2(x^{(k)} - x)\\
\Rightarrow &||x^{(k+1)} - x|| = ||G^2(x^{(k)} - x)||\\
\Rightarrow &||x^{(k+1)} - x|| \leq ||G^2||\ ||(x^{(k)} - x)||\\
&\leq ||G^4||\ ||(x^{(k-1)} - x)||\\
&\leq ... \leq ||G^{2k+2}||\ ||(x^{(0)} - x)||\\
&= ||G||^{2k+2}\ ||(x^{(0)} - x)||
\end{aligned}$$

Again, considering any p^{th} RGJ,

$$\begin{aligned}
&x^{(k+1)} = G^{p+1}x^{(k)} + K_{p+1}\\
\Rightarrow &x^{(k+1)} - x = G^{p+1}x^{(k)} - x + K_{p+1}\\
\Rightarrow &x^{(k+1)} - x = G^{p+1}(x^{(k)} - x) + G^{p+1}x + K_{p+1} - x\\
\Rightarrow &x^{(k+1)} - x = G^{p+1}(x^{(k)} - x)
\end{aligned}$$

$$\Rightarrow ||x^{(k+1)} - x|| = ||G^{p+1}(x^{(k)} - x)||$$

$$\begin{aligned}\Rightarrow ||x^{(k+1)} - x|| &\leq ||G^{p+1}||\ ||(x^{(k)} - x)|| \\ &\leq ||G^{2(p+1)}||\ ||(x^{(k-1)} - x)|| \\ &\leq \ldots \leq ||G^{(p+1)(k+1)}||\ ||(x^{(0)} - x)|| \\ &= ||G||^{(p+1)(k+1)}\ ||(x^{(0)} - x)||\end{aligned}$$

According to the coefficient of the above inequalities,

We have,

$$||G||^{(p+1)(k+1)} \leq \ldots \leq ||G||^{2(k+1)} \leq ||G||^{k+1} \qquad \text{(From 7)}$$

The p^{th} Refinement of the Generalized Jacobi method achieves faster convergence than the original, with the $(p + 1)^{th}$ Refinement outperforming the p^{th} for any initial guess $x^{(0)}$.

Here, we present two illustrative numerical examples to validate the accuracy and effectiveness of our established theorems.

Example 4.1. (Audu *et al.*, 2023) Consider an example of a 9 × 9 matrix, where the matrix is SDD and with an error tolerance 10^{-4}.

$$\begin{aligned}&4x_1 - x_2 = 20\\ &-x_1 + 4x_2 - x_3 - x_5 = 20\\ &-x_2 + 4x_3 - x_5 = 0\\ &-x_2 + 4x_4 - x_5 - x_7 = 20\\ &-x_3 - x_4 + 4x_5 - x_6 - x_8 = 0\\ &-x_5 + 4x_6 - x_9 = 0\\ &-x_4 + 4x_7 - x_8 = 30\\ &-x_5 - x_7 + 4x_8 - x_9 = 10\\ &-x_6 - x_8 + 4x_9 = 10\end{aligned}$$

Now, solving this system in MATLAB for different iterative methods with the error tolerance 10^{-4} and $(0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ as an initial approximation, we compare the number of iterations, their spectral radius, and solutions for different methods as shown in the tables below.

Table 1: Comparison with No. of Iterations and Spectral Radius.

Methods	No. of Iterations	Spectral Radius
GJ	14	0.5177
RGJ	9	0.2681
2^{nd} RGJ	6	0.1388
3^{rd} RGJ	5	0.0719
4^{th} RGJ	4	0.0372
5^{th} RGJ	4	0.0193
6^{th} RGJ	4	0.0100
7^{th} RGJ	3	0.0052
8^{th} RGJ	3	0.0027
9^{th} RGJ	3	0.0014
10^{th} RGJ	3	7.1653×10^{-4}
11^{th} RGJ	3	3.7098×10^{-4}
12^{th} RGJ	3	1.9207×10^{-4}
13^{th} RGJ	3	9.9442×10^{-5}
14^{th} RGJ	2	5.1485×10^{-5}
15^{th} RGJ	2	2.6656×10^{-5}

Table 2: Numerical Results for GJ, RGJ, SRGJ, TRGJ,. . . , 15^{th} RGJ with m = 1.

Method	n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_4^{(n)}$	$x_5^{(n)}$	$x_6^{(n)}$	$x_7^{(n)}$	$x_8^{(n)}$	$x_9^{(n)}$
GJ	0	0	0	0	0	0	0	0	0	0
	1	6.7857	7.1429	1.7857	5.3571	1.4286	0.3571	8.9286	5.7143	3.9286
	2	6.9133	7.6531	2.2704	10.2679	5.0000	2.2321	10.4719	6.5306	4.2219
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	14	7.4260	9.7042	4.2189	12.4255	7.1731	3.1712	12.8254	8.8772	5.5120
RGJ	0	0	0	0	0	0	0	0	0	0
	1	6.9133	7.6531	2.2704	10.2679	5.0000	2.2321	10.4719	6.5306	4.2219
	2	7.2803	9.1212	3.6651	11.8965	6.5816	2.9121	12.2918	8.2510	5.1728
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	9	7.4262	9.7049	4.2197	12.4262	7.1739	3.1716	12.8260	8.8780	5.5124
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(Table 2 Contd...)

(...Contd. Table 2)

	0	0	0	0	0	0	0	0	0	0
15^{th}RGJ	1	7.4262	9.7048	4.2195	12.4260	7.1737	3.1715	12.8259	8.8779	5.5123
	2	7.4262	9.7050	4.2197	12.4262	7.1739	3.1716	12.8261	8.8781	5.5124

Table 1 demonstrates the iteration counts for different methods. The Generalized Jacobi(GJ) method requires 14 iterations, the Refinement of Generalized Jacobi(RGJ) method needs 9 iterations, the Second Refinement of Generalized Jacobi(SRGJ) method takes 6 iterations, and so up to the 15^{th} Refinement of Generalized Jacobi method which takes only 2 iterations. Table 2 shows the solutions obtained by these respective methods. This indicates that increasing the refinement levels in the Generalized Jacobi methods reduces both spectral radii and the number of iterations.

Example 4.2. (Meligy and Youssef, 2022) Consider an example of an 8 × 8 matrix, where the matrix is SDD and with an error tolerance 10^{-4}.

$$4.2x_1 - x_3 - x_4 - x_7 - x_8 = 6.2$$
$$-x_1 + 4.2x_2 - x_4 - x_5 - x_8 = 5.4$$
$$-x_1 - x_2 + 4.2x_3 - x_5 - x_6 = -9.20$$
$$-x_2 - x_3 + 4.2x_4 - x_6 - x_7 = 0.00$$
$$-x_3 - x_4 + 4.2x_5 - x_7 - x_8 = 6.20$$
$$-x_1 - x_4 - x_5 + 4.2x_6 - x_8 = 1.20$$
$$-x_1 - x_2 - x_5 - x_6 + 4.2x_7 = -13.4$$
$$-x_2 - x_3 - x_6 - x_7 + 4.2x_8 = 4.20$$

Utilizing MATLAB, we approach solving this system through several iterative methods. Starting with an initial guess of $(0, 0, 0, 0, 0, 0, 0, 0)^T$ and setting an error tolerance of 10^{-4}, we analyze and compare the performance in terms of iteration counts and spectral radii, as depicted in the following tables.

Table 3: Comparison with No. of Iterations and Spectral Radius.

Methods	No. of Iterations	Spectral Radius
GJ	90	0.9398
RGJ	51	0.8833
2^{nd} RGJ	37	0.8302
3^{rd} RGJ	29	0.7802
4^{th} RGJ	24	0.7333
5^{th} RGJ	21	0.6892
6^{th} RGJ	18	0.6477
7^{th} RGJ	16	0.6088
8^{th} RGJ	15	0.5721
9^{th} RGJ	14	0.5377
10^{th} RGJ	13	0.5054
11^{th} RGJ	12	0.4750
12^{th} RGJ	11	0.4464
13^{th} RGJ	10	0.4195
14^{th} RGJ	10	0.3943
15^{th} RGJ	9	0.3706

Table 4: Numerical Results for GJ, RGJ, SRGJ, TRGJ,. . . , 15^{th} RGJ with m = 1.

Method	n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_4^{(n)}$	$x_5^{(n)}$	$x_6^{(n)}$	$x_7^{(n)}$	$x_8^{(n)}$
GJ	0	0	0	0	0	0	0	0	0
	1	1.4762	1.6372	−1.8007	−0.4287	1.3741	0.6129	−3.0446	0.2751
	2	0.2860	1.6444	−0.9744	−0.4212	0.2878	0.6691	−1.9627	0.6397
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	90	0.9985	1.9985	−1.0015	−0.0015	0.9985	0.9985	−2.0015	0.9985
RGJ	0	0	0	0	0	0	0	0	0
	1	0.2860	1.6444	−0.9744	−0.4212	0.2878	0.6691	−1.9627	0.6397
	2	0.6223	1.7449	−1.2265	−0.3639	0.6015	0.7318	−2.2315	0.6359
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	51	0.9993	1.9993	−1.0007	−0.0007	0.9993	0.9993	−2.0007	0.9993
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(Table 4 Contd...)

(...Contd. Table 4)

15^{th} *RGJ*	0	0	0	0	0	0	0	0	0
	1	0.8497	1.8516	−1.1486	−0.1481	0.8520	0.8515	−2.1486	0.8519
	2	0.9443	1.9450	−1.0550	−0.0549	0.9451	0.9450	−2.0550	0.9451
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	9	0.9999	1.9999	−1.0001	−0.00001	0.9999	0.9999	−2.0001	0.9999

Table 3 represents the number of iterations and the spectral radius of different refinement methods and it observes that GJ takes 90 iterations to converge, which decreases as the application of refinement and it took only 9 iterations in the case of 15^{th} RGJ method. So from both tables, it is clear that increasing the refinement level in the Generalized Jacobi methods leads to a reduction in both the spectral radii and the number of iterations required for convergence. So, any $(p + 1)^{th}$ Refinement is better than that of the p^{th} Refinement of Generalized Jacobi method.

We have demonstrated results up to the 15^{th} RGJ method in the above examples. However, the same approach can be extended to any higher-order RGJ methods, and the observed trends regarding convergence and spectral radii will remain true.

CONCLUSION

In summary, our study focused on the comparative analysis of the p^{th} Refinement of Generalized Jacobi method, covering derivation, convergence characteristics, and numerical performance assessment. Initially, we derived the Third Refinement of Generalized Jacobi method and formulated a comprehensive expression applicable to any p^{th} Refinement. Our investigation involved rigorous theoretical scrutiny, including proofs of convergence for the p^{th} Refinement method across various matrix types such as SDD and M-matrices. Our research revealed a significant insight: the convergence rate of the $(p + 1)^{th}$ Refinement exceeds that of the p^{th} Refinement once the Generalized Jacobi method achieves convergence. This observation underscores the accelerated convergence potential offered by higher-order Refinements within the Generalized Jacobi method. To support our theoretical assertions, we conducted numerical experiments, demonstrating the superior convergence rate of the $(p + 1)^{th}$ Refinement compared to its predecessor, the p^{th} Refinement.

Overall, our findings highlight the efficacy and theoretical advantages of higher-order Refinements in improving the convergence properties of the Generalized Jacobi method. These results contribute valuable insights to the field of iterative methods for solving linear systems of equations, with potential applications in enhancing computational efficiency and accuracy across various scientific and engineering domains.

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